RELATIONSHIP BETWEEN THE HYERS–ULAM STABILITY AND THE MOORE–PENROSE INVERSE

QIANGLIAN HUANG[1*] AND MOHAMMAD SAL MOSLEHIAN[2]

ABSTRACT. In this paper, we establish a link between the Hyers–Ulam stability and the Moore–Penrose inverse, that is, a closed operator has the Hyers–Ulam stability if and only if it has a bounded Moore–Penrose inverse. Meanwhile, the stability constant can be determined in terms of the Moore–Penrose inverse. Based on this result, some conditions for the perturbed operators having the Hyers–Ulam stability are obtained and the Hyers–Ulam stability constant is expressed explicitly in the case of closed operators. In the case of the bounded linear operators we obtain some characterizations for the Hyers–Ulam stability constants to be continuous. As an application, we give a characterization for the Hyers–Ulam stability constants of the semi-Fredholm operators to be continuous.

1. INTRODUCTION AND PRELIMINARIES

More than a half century ago, Ulam [27] proposed the first stability problem concerning group homomorphisms, which was partially solved by Hyers [13] in the framework of Banach spaces. Later, Aoki [1] proved the stability of the additive mapping and Th.M. Rassias [24] investigated the stability of the linear mapping for mappings f when the norm of the Cauchy difference $f(x + y) - f(x) - f(y)$ is bounded by the expression $\varepsilon(\|x\|^p + \|y\|^p)$ for some $\varepsilon \geq 0$ and some $0 \leq p < 1$. J.M. Rassias [23] considered the same problem with $\varepsilon(\|x\|^p \|y\|^p)$. A large number of papers have been published in connection with various generalizations of Hyers–Ulam theorem in several wide frameworks. In particular, it is nearly related to the notion of perturbation [5, 19] and geometry of Banach spaces [7, 26]. The interested reader is referred to books [6, 14, 15, 20] and references therein.

In 2003, Miura, Miyajima and Takahasi [17, 18] investigated the notion of the Hyers–Ulam stability of a mapping between normed linear spaces and obtained some stability results for particular linear differential operators. Takagi, Miura and Takahasi [25] considered the Hyers–Ulam stability of bounded linear operators

in Banach spaces. After then, Hirasawa and Miura [9] gave some necessary and sufficient conditions under which a closed operator in a Hilbert space has the Hyers–Ulam stability. Moslehian and Sadeghi [21] studied the Hyers–Ulam stability of T-bounded operators. They also discussed the best constant of Hyers–Ulam stability. In the sequel, we need some terminology.

Definition 1.1. Let X, Y be normed linear spaces and let T be a (not necessarily linear) mapping from X into Y. We say that T has the Hyers–Ulam stability if there exists a constant $K > 0$ with the property: (a) For any y in the range $R(T)$ of T, $\varepsilon > 0$ and x in the Domain $D(T)$ of T with $\|Tx - y\| \leq \varepsilon$, there exists an element $x_0 \in D(T)$ such that $Tx_0 = y$ and $\|x - x_0\| \leq K\varepsilon$. We call such $K > 0$ a Hyers–Ulam stability constant for T and denote by K_T the infimum of all Hyers–Ulam stability constants for T. If K_T is a Hyers–Ulam stability constant for T, then K_T is called the Hyers–Ulam stability constant for T.

Roughly speaking, if T has the Hyers–Ulam stability, then to each ε-approximate solution x of the equation $Tx = y$ there corresponds an exact solution x_0 of the equation in a K_ε-neighborhood of x; see [9].

Remark 1.2. [9, 21] If T is linear then condition (a) is equivalent to:
(b) For any $\varepsilon > 0$ and $x \in D(T)$ with $\|Tx\| \leq \varepsilon$, there exists $x_0 \in D(T)$ such that $Tx_0 = 0$ and $\|x - x_0\| \leq K\varepsilon$.
If we denote the null space of T by $N(T)$, then the condition (b) is equivalent to:
(c) For any $x \in D(T)$, there exists $x_0 \in N(T)$ such that $\|x - x_0\| \leq K\|Tx\|$.

Let X, Y be Banach spaces. Let $L(X, Y)$, $C(X, Y)$ and $B(X, Y)$ denote the linear space of all linear operators, the homogeneous set of all closed linear operators with a dense domain and the Banach space of all bounded linear operators from X into Y, respectively. The identity operator is denoted by I. Let us introduce the reduced minimum modulus of closed linear operators.

Definition 1.3. [16] The reduced minimum modulus of $T \in C(X, Y)$ is defined by

$$\gamma(T) = \inf\{\|Tx\| : \quad x \in D(T) \text{ with } d(x, N(T)) := \inf_{z \in N(T)} d(x, z) = 1\}.$$

It is easy to see that $\gamma(T) = \sup\{\alpha \geq 0 : \|Tx\| \geq \alpha\, d(x, N(T)),\ x \in D(T)\}$ and if X and Y are Hilbert spaces, then

$$\gamma(T) = \inf\{\|Tx\| : \quad x \in D(T) \cap N(T)^\perp \text{ with } \|x\| = 1\},$$

where $\perp$ denotes the orthogonal complement in Hilbert spaces; see also [4].

Theorem 1.4. [9] *Let X, Y be Hilbert spaces and $T \in C(X, Y)$. Then T has the Hyers–Ulam stability if and only if it has closed range. In this case, $K_T = \gamma(T)^{-1}$.*

Let us introduce the notion of a generalized inverse (see e.g. [2]) and that of the Moore–Penrose inverse of a closed operator.

Definition 1.5. [2] An operator $T \in C(X, Y)$ possesses a (bounded) generalized inverse if there exists an operator $S \in B(Y, X)$ such that $R(S) \subseteq D(T)$ and (1) $TSTx = Tx$ for all $x \in D(T)$; (2) $STSy = Sy$ for all $y \in Y$; (3) ST is continuous. We denote a generalized inverse of T by T^+.

In general, the generalized inverse need not exist and is not unique even if it exists. We need the following lemma concerning the existence of generalized inverses.

Lemma 1.6. [22] *(a) Let $T \in C(X, Y)$. Suppose that $N(T)$ has a topological complement $N(T)^c$ in X and $\overline{R(T)}$ has a topological complement $\overline{R(T)}^c$ in Y, i.e.,*

$$X = N(T) \oplus N(T)^c \quad and \quad Y = \overline{R(T)} \oplus \overline{R(T)}^c.$$

Let P denote the projector of X onto $N(T)$ along $N(T)^c$ and Q denote the projector of Y onto $\overline{R(T)}$ along $\overline{R(T)}^c$. Then there is a unique $S \in C(Y, X)$ satisfying: 1) $TST = T$ on $D(T)$; 2) $STS = S$ on $D(S)$; 3) $ST = I - P$ on $D(T)$ and 4) $TS = Q$ on $D(S)$, where $D(S) = R(T) + \overline{R(T)}^c$.
(b) Under the assumptions of part (a), S is bounded if and only if $R(T)$ is closed. In this case, S is a bounded generalized inverse of T with $D(S) = Y$, $N(S) = R(T)^c$ and $R(S) = D(T) \cap N(T)^c$.

Definition 1.7. [22] Let X, Y be Hilbert spaces and $T \in C(X, Y)$. If the topological decompositions in Lemma 1.6 are orthogonal, i.e.,

$$X = N(T) \dotplus N(T)^\perp \quad and \quad Y = \overline{R(T)} \dotplus R(T)^\perp,$$

where $\dotplus$ denotes the orthogonal direct sum, then the corresponding generalized inverse is usually called the Moore–Penrose inverse of T. In this case, the operators P and Q in Lemma 1.6 are orthogonal projectors. The Moore–Penrose inverse of T is always denoted by $T^\dagger$.

Remark 1.8. The operator $T \in C(X, Y)$ has a generalized inverse $T^+ \in B(Y, X)$ if and only if

$$X = N(T) \oplus N(T)^c \quad and \quad Y = R(T) \oplus R(T)^c.$$

In this case, it follows from the closed graph theorem that the operator TT^+ is a projector from Y onto $R(T)$ such that $N(TT^+) = N(T^+)$ and $R(TT^+) = R(T)$. Meanwhile, by the condition (3) in Definition 1.5, T^+T can be extended uniquely to a projector from X onto $\overline{R(T^+)}$ with the null space $N(T)$ and the range $\overline{R(T^+)}$.

It is well known that the perturbation analysis of Moore–Penrose inverses and generalized inverses in Hilbert and Banach spaces are very important in practical applications of operator theory and has been widely studied; cf. [3, 4, 8, 10, 11, 12, 22, 28, 29]. Recently, the perturbation of generalized inverses for linear operators in Hilbert spaces or Banach spaces has been studied in [4, 8, 10, 11, 12, 28, 29]. To achieve our results, we need the concept of $T-$boundedness as follows.

Definition 1.9. [16] Let T and P be linear operators with the same domain space such that $D(T) \subseteq D(P)$ and

$$\|Px\| \le a\|x\| + b\|Tx\| \qquad (x \in D(T)),$$

where a, b are nonnegative constants. Then we say P is relatively bounded with respect to T or simply T-bounded and the greatest lower bound of all possible constants b will be called the relative bound of P with respect to T or simply the T-bound.

Theorem 1.10. [12] *Let X, Y be Banach spaces and let $T \in C(X,Y)$ with a bounded generalized inverse $T^+ \in B(Y,X)$. Let $\delta T \in L(X,Y)$ be $T-$bounded with constants a, b. If $a\|T^+\| + b\|TT^+\| < 1$, then the following statements are equivalent:*
(1) $B := T^+(I + \delta TT^+)^{-1} : Y \to X$ is a bounded generalized inverse of $\overline{T} := T + \delta T$;
(2) $(I + \delta TT^+)^{-1} R(\overline{T}) = R(T)$;
(3) $(I + \delta TT^+)^{-1}\overline{T}$ maps $N(T)$ into $R(T)$;
(4) $R(\overline{T}) \cap N(T^+) = \{0\}$.
   Moreover, if one of the conditions above is true, then $\overline{T}$ is a closed operator and its range $R(\overline{T})$ is closed.

In this paper, we use the expression and the stability characterization of the Moore–Penrose inverse to investigate the condition for the perturbed operators to have the Hyers–Ulam stability and the condition in order that the Hyers–Ulam stability constant be continuous. In Section 2, we first establish a relationship between the Hyers–Ulam stability and the Moore–Penrose inverse, that is, a closed operator has the Hyers–Ulam stability if and only if it has the bounded Moore–Penrose

inverse. Meanwhile, the stability constant is determined in terms of the Moore–Penrose inverse. Utilizing this result, we give some sufficient conditions for the perturbed operators having the Hyers–Ulam stability and give an explicit expression of the Hyers–Ulam stability constant. In the case of bounded linear operators, some sufficient and necessary conditions for the Hyers–Ulam stability constants to be continuous are also provided in Section 3. In the end, as an application, we give a characterization for the Hyers–Ulam stability constants of the semi-Fredholm operators to be continuous.

2. The case of closed linear operators

The following lemma is proved by a straightforward verification of the conditions of the definition of Moore–Penrose inverse.

Lemma 2.1. [12] *Let X, Y be Hilbert spaces and $T \in C(X, Y)$ with a generalized inverse $T^+ \in B(Y, X)$, then T has the bounded Moore–Penrose inverse $T^\dagger$ and*

$$T^\dagger = [I - P^\perp_{N(T)}]T^+ P^\perp_{R(T)}.$$

Theorem 2.2. *Let X, Y be Hilbert spaces and $T \in C(X, Y)$. Then the following statements are equivalent:*
(1) T has the Hyers–Ulam stability;
(2) T has the bounded Moore–Penrose inverse $T^\dagger$;
(3) T has a bounded generalized inverse T^+;
(4) T has closed range.
Moreover, if one of the conditions above is true, then $D(T^\dagger) = Y$, $N(T^\dagger) = R(T)^\perp$, $R(T^\dagger) = D(T) \cap N(T)^\perp$ and

$$K_T = \|T^\dagger\| = \gamma(T)^{-1}.$$

Proof. Note that $T|_{N(T)^\perp} : D(T) \cap N(T)^\perp \to R(T)$ is invertible and $T^\dagger$ is defined by

$$T^\dagger y = (T|_{N(T)^\perp})^{-1}Qy \qquad (y \in R(T) + R(T)^\perp),$$

where Q is the orthogonal projector of Y onto $\overline{R(T)}$ along $R(T)^\perp$. Then $T^\dagger$ is a densely defined closed operator with $D(T^\dagger) = R(T) + R(T)^\perp$ and $R(T^\dagger) = D(T) \cap N(T)^\perp$.

(2) $\Rightarrow$ (1). If T has the bounded Moore–Penrose inverse $T^\dagger$, then for all $x \in D(T)$, $(I - T^\dagger T)x \in N(T)$ and

$$\|x - (I - T^\dagger T)x\| = \|T^\dagger T x\| \le \|T^\dagger\|\|Tx\|.$$

This means that T has the Hyers–Ulam stability and $K_T \le \|T^\dagger\|$. Assume that K is a Hyers–Ulam stability constant for T, i.e., for all $x \in D(T)$, there exists $x_0 \in N(T)$ such that $\|x - x_0\| \le K\|Tx\|$. Then for all $y \in Y$, $T^\dagger y \in D(T) \cap N(T)^\perp$ and there exists $x_1 \in N(T)$ such that $\|T^\dagger y - x_1\| \le K\|TT^\dagger y\|$. Since $T^\dagger y \perp x_1$, we get

$$\|T^\dagger y\| \le \|T^\dagger y - x_1\| \le K\|TT^\dagger y\| \le K\|y\|. \tag{2.1}$$

Hence $K \ge \|T^\dagger\|$ and thus $K_T \ge \|T^\dagger\|$. Therefore $K_T = \|T^\dagger\|$.

$(1) \Rightarrow (2)$. If T has the Hyers–Ulam stability, then by (2.1), we know that the Moore–Penrose inverse $T^\dagger$ is bounded.

$(2) \Leftrightarrow (3)$. From Lemma 2.1, we can see $(3) \Rightarrow (2)$. Since the Moore–Penrose inverse $T^\dagger$ is also a generalized inverse, we can get $(2) \Rightarrow (3)$.

$(4) \Rightarrow (2)$. If $R(T)$ is closed, then $D(T^\dagger) = Y$ and by the Closed Graph Theorem, $T^\dagger$ is bounded.

$(2) \Rightarrow (4)$. If $T^\dagger$ is bounded, since $T^\dagger$ is a densely defined closed operator, we get $D(T^\dagger) = Y$, i.e, $Y = R(T) + R(T)^\perp$. This implies that $R(T)$ is closed. In the following, we shall show $\gamma(T) = \|T^\dagger\|^{-1}$. In fact, for all $x \in D(T)$, we have $(I - T^\dagger T)x \in N(T)$ and

$$d(x, N(T)) \le \|x - (I - T^\dagger T)x\| = \|T^\dagger Tx\| \le \|T^\dagger\|\|Tx\|.$$

Then $\gamma(T) \ge \|T^\dagger\|^{-1}$. Since $\gamma(T) \le \|Tx\|$ for all $x \in D(T) \cap N(T)^\perp$ with $\|x\| = 1$, we get for all $y \in Y$ satisfying $\|T^\dagger y\| = 1$,

$$\gamma(T) \le \|TT^\dagger y\| = \|Qy\| \le \|y\|.$$

Hence for all $y \in Y$ with $T^\dagger y \neq 0$, $\gamma(T) \le \frac{\|y\|}{\|T^\dagger y\|}$. Thus

$$\gamma(T) \le \inf\left\{ \frac{\|y\|}{\|T^\dagger y\|} : y \in Y, T^\dagger y \neq 0 \right\} = \left(\sup\left\{ \frac{\|T^\dagger y\|}{\|y\|} : y \in Y \right\} \right)^{-1} = \|T^\dagger\|^{-1}$$

Therefore $\gamma(T) = \|T^\dagger\|^{-1}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Applying Theorem 2.2, we get [9, Theorem 3.1].

Corollary 2.3. [9] *Let X, Y be Hilbert spaces and $T \in C(X, Y)$. Then T has the Hyers–Ulam stability if and only if T has closed range. Moreover, in this case, $K_T = \gamma(T)^{-1}$.*

In the following, we shall use the expressions and stability characterizations of the Moore–Penrose inverse to investigate the Hyers–Ulam stability of closed operators.

We need the following lemma, which can be proved by using the fact that $P_M P_M^*[I - (P_M - P_M^*)^2] = [I - (P_M - P_M^*)^2]P_M P_M^*$.

Lemma 2.4. [12] *Let X be a Hilbert space and M be a closed linear subspace of X. Let $P_M : X \to M$ be a (not necessarily selfadjoint) projector from X onto M, then the orthogonal projector $P_M^\perp$ from X onto M can be expressed by*

$$P_M^\perp = P_M P_M^*[I - (P_M - P_M^*)^2]^{-1} = [I - (P_M - P_M^*)^2]^{-1} P_M P_M^*.$$

Utilizing Lemma 2.1 with Lemma 2.4, we can get the following lemma.

Lemma 2.5. *Let $T \in C(X, Y)$ with a bounded generalized inverse $T^+ \in B(Y, X)$, then T has the bounded Moore–Penrose inverse $T^\dagger$ and*

$$T^\dagger = \{I - [(T^+T)^{**} - (T^+T)^*]^2\}^{-1}(T^+T)^*T^+(TT^+)^*\{I - [TT^+ - (TT^+)^*]^2\}^{-1}$$

Proof. Since TT^+ is a projector from Y onto $R(T)$, it follows from Lemma 2.4 that

$$P_{R(T)}^\perp = TT^+(TT^+)^*\{I - [TT^+ - (TT^+)^*]^2\}^{-1}.$$

Noting that $I - T^+T$ is a bounded projector from $D(T)$ onto $N(T)$ and $D(T)$ is dense in X, one can verify that $(I - T^+T)^*$ is defined on the whole space Y. It follows from the Closed Graph Theorem that $(I - T^+T)^*$ is bounded. Hence $P_{N(T)} = (I - T^+T)^{**}$, which is exactly the unique norm-preserving extension to whole space X of $I - T^+T$. Thus $P_{N(T)}^* = (I - T^+T)^*$ and

$$
\begin{aligned}
&I - P_{N(T)}^\perp \\
=\ & I - [I - (P_{N(T)} - P_{N(T)}^*)^2]^{-1} P_{N(T)} P_{N(T)}^* \\
=\ & [I - (P_{N(T)} - P_{N(T)}^*)^2]^{-1}[I - (P_{N(T)} - P_{N(T)}^*)^2 - P_{N(T)} P_{N(T)}^*] \\
=\ & [I - (P_{N(T)} - P_{N(T)}^*)^2]^{-1}(I - P_{N(T)}^*)(I - P_{N(T)}) \\
=\ & \{I - [(I - T^+T)^{**} - (I - T^+T)^*]^2\}^{-1}[I - (I - T^+T)^*][I - P_{N(T)}] \\
=\ & \{I - [(T^+T)^{**} - (T^+T)^*]^2\}^{-1}(T^+T)^*[I - P_{N(T)}].
\end{aligned}
$$

Therefore, by $P_{N(T)}|_{D(T)} = I - T^+T$ and $R(T^+) \subset D(T)$, we obtain

$$
\begin{aligned}
T^\dagger &= [I - P_{N(T)}^\perp]T^+ P_{R(T)}^\perp \\
&= \{I - [(T^+T)^{**} - (T^+T)^*]^2\}^{-1}(T^+T)^*[I - P_{N(T)}] \\
&\quad T^+(TT^+)(TT^+)^*\{I - [TT^+ - (TT^+)^*]^2\}^{-1} \\
&= \{I - [(T^+T)^{**} - (T^+T)^*]^2\}^{-1}(T^+T)^*T^+(TT^+)^*\{I - [TT^+ - (TT^+)^*]^2\}^{-1}.
\end{aligned}
$$

$\square$

Lemma 2.6. *Let $T \in C(X,Y)$ have a generalized inverse $T^+ \in B(Y,X)$ and $\delta T \in L(X,Y)$ be $T-$bounded with constants a, b. If $a\|T^+\|+b\|TT^+\| < 1$, then $\overline{T} = T+\delta T$ is closed and*

$$B = T^+(I + \delta TT^+)^{-1} : \; Y \to X$$

satisfies $B\overline{T}B = B$ on Y, $R(B) = R(T^+)$ and $N(B) = N(T^+)$.

Proof. It follows from Theorem 1.10 that $\overline{T}$ is closed. Since

$$\begin{aligned}
\|\delta TT^+y\| &\leq a\|T^+y\| + b\|TT^+y\| \\
&\leq (a\|T^+\| + b\|TT^+\|)\|y\|,
\end{aligned}$$

for all $y \in Y$, we get $\|\delta TT^+\| \leq a\|T^+\| + b\|TT^+\| < 1$. By the celebrated Banach Lemma, the inverse of $I+\delta TT^+$ exists and $(I+\delta TT^+)^{-1} \in B(Y)$. Hence $B = T^+(I+\delta TT^+)^{-1} : Y \to X$ is a bounded linear operator. It is easy to verify $R(B) = R(T^+)$ and $N(B) = N(T^+)$. To the end, we need to show $B\overline{T}B = B$ on Y. Indeed, $R(B) = R(T^+) \subseteq D(T) = D(\overline{T})$. Since

$$(I + \delta TT^+)TT^+ = TT^+ + \delta TT^+ = (T + \delta T)T^+ = \overline{T}T^+,$$

we have $TT^+ = (I + \delta TT^+)^{-1}\overline{T}T^+$ and therefore,

$$\begin{aligned}
B\overline{T}B &= T^+(I + \delta TT^+)^{-1}\overline{T}T^+(I + \delta TT^+)^{-1} \\
&= T^+TT^+(I + \delta TT^+)^{-1} = T^+(I + \delta TT^+)^{-1} = B.
\end{aligned}$$

$\square$

In the next theorem some conditions are given for $\overline{T}$ to have the Hyers–Ulam stability as well as the Hyers–Ulam stability constant $K_{\overline{T}}$ is explicitly expressed.

Theorem 2.7. *Let X, Y be Hilbert spaces and let $T \in C(X,Y)$ have the Hyers–Ulam stability. Let $T^+ \in B(Y,X)$ be a bounded generalized inverse of T and $\delta T \in L(X,Y)$ be $T-$bounded with constants a, b. If $a\|T^+\| + b\|TT^+\| < 1$, then the following statements are equivalent:*
(1) $B = T^+(I+\delta TT^+)^{-1} : \; Y \to X$ is a bounded generalized inverse of $\overline{T} = T+\delta T$;
(2) $(I + \delta TT^+)^{-1}R(\overline{T}) = R(T)$;
(3) $(I + \delta TT^+)^{-1}\overline{T}$ maps $N(T)$ into $R(T)$;
(4) $R(\overline{T}) \cap N(T^+) = \{0\}$.

Moreover, if one of the conditions above is true, then $R(\overline{T})$ is closed, $\overline{T}$ has the Hyers–Ulam stability and $K_{\overline{T}} = \|\overline{T}^{\dagger}\|$, where

$$
\begin{aligned}
\overline{T}^{\dagger} &= \{I - [(T^{+}(I + \delta TT^{+})^{-1}\overline{T})^{**} - (T^{+}(I + \delta TT^{+})^{-1}\overline{T})^{*}]^{2}\}^{-1} \\
&\quad [T^{+}(I + \delta TT^{+})^{-1}\overline{T}]^{*}T^{+}(I + \delta TT^{+})^{-1}[\overline{T}T^{+}(I + \delta TT^{+})^{-1}]^{*} \\
&\quad \{I - [\overline{T}T^{+}(I + \delta TT^{+})^{-1} - (\overline{T}T^{+}(I + \delta TT^{+})^{-1})^{*}]^{2}\}^{-1}.
\end{aligned}
$$

Proof. From Theorem 1.10, we can see the equivalence. If one of the conditions is true, then $R(\overline{T})$ is closed and $T^{+}(I + \delta TT^{+})^{-1} : Y \to X$ is a bounded generalized inverse of $T + \delta T$. By Lemma 2.5, we can get what we desired. □

Remark 2.8. It should be noted that under any one of the conditions in Theorem 2.7, the Hyers–Ulam stability constant $K_{\overline{T}}$ is continuous at T if δT is bounded and $\|\delta T\| \to 0$. In fact,

$$
\begin{aligned}
T^{+}(I + \delta TT^{+})^{-1}\overline{T} &= (I + T^{+}\delta T)^{-1}T^{+}\overline{T} \\
&= (I + T^{+}\delta T)^{-1}T^{+}T + (I + T^{+}\delta T)^{-1}T^{+}\delta T \to T^{+}T
\end{aligned}
$$

and

$$
\begin{aligned}
&\overline{T}T^{+}(I + \delta TT^{+})^{-1} \\
&= TT^{+}(I + \delta TT^{+})^{-1} + \delta TT^{+}(I + \delta TT^{+})^{-1} \to TT^{+},
\end{aligned}
$$

as $\|\delta T\| \to 0$.

Corollary 2.9. *Let X, Y be Hilbert spaces and let $T \in C(X, Y)$ have the Hyers–Ulam stability. Let $T^{+} \in B(Y, X)$ be a bounded generalized inverse of T and $\delta T \in L(X, Y)$ be $T-$bounded with constants a, b. If $a\|T^{+}\| + b\|TT^{+}\| < 1$ and $N(T) \subseteq N(\delta T)$, then $\overline{T} = T + \delta T$ has the Hyers–Ulam stability and $K_{\overline{T}} = \|\overline{T}^{\dagger}\|$, where*

$$
\begin{aligned}
\overline{T}^{\dagger} &= \{I - [(T^{+}T)^{**} - (T^{+}T)^{*}]^{2}\}^{-1}(T^{+}T)^{*}T^{+}(I + \delta TT^{+})^{-1}[\overline{T}T^{+}(I + \delta TT^{+})^{-1}]^{*} \\
&\quad \{I - [\overline{T}T^{+}(I + \delta TT^{+})^{-1} - ((\overline{T}T^{+}(I + \delta TT^{+})^{-1})^{*}]^{2}\}^{-1}.
\end{aligned}
$$

Proof. We first show that $N(\overline{T}) = N(T)$. In fact, by $N(T) \subseteq N(\delta T)$, we have $N(T) \subseteq N(\overline{T})$ as well as $\delta T(I - T^{+}T) = 0$, whence $\delta T = \delta TT^{+}T$. If $x \in N(\overline{T})$, then

$$0 = \overline{T}x = Tx + \delta Tx = Tx + \delta TT^{+}Tx = (I + \delta TT^{+})Tx.$$

Since $I + \delta TT^{+}$ is invertible, $Tx = 0$. Hence $N(\overline{T}) = N(T)$. Thus $P^{\perp}_{N(\overline{T})} = P^{\perp}_{N(T)}$ and $P^{\perp}_{N(\overline{T})}|_{D(T)} = I - \{I - [(I - T^{+}T)^{**} - (I - T^{+}T)^{*}]^{2}\}^{-1}[I - (I - T^{+}T)^{*}]T^{+}T$. Therefore, by Lemma 2.5, we can get what we desired. □

Corollary 2.10. *Let X, Y be Hilbert spaces and let $T \in C(X,Y)$ have the Hyers–Ulam stability. Let $T^+ \in B(Y,X)$ be a bounded generalized inverse of T and $\delta T \in L(X,Y)$ be T−bounded with constants a, b. If $a\|T^+\| + b\|TT^+\| < 1$ and $R(\delta T) \subseteq R(T)$, then $\overline{T} = T + \delta T$ has the Hyers–Ulam stability and $K_{\overline{T}} = \|\overline{T}^\dagger\|$, where*

$$\overline{T}^\dagger = \{I - [(T^+(I + \delta TT^+)^{-1}\overline{T})^{**} - (T^+(I + \delta TT^+)^{-1}\overline{T})^*]^2\}^{-1}$$
$$[T^+(I + \delta TT^+)^{-1}\overline{T}]^*T^+(I + \delta TT^+)^{-1}(TT^+)^*\{I - [TT^+ - (TT^+)^*]^2\}^{-1}.$$

Proof. We first show $R(\overline{T}) = R(T)$. In fact, by $R(\delta T) \subseteq R(T)$, we have $R(\overline{T}) \subseteq R(T)$ and also $(I - TT^+)\delta T = 0$, which implies $\delta T = TT^+\delta T$. Then

$$\overline{T}T^+ = (T + \delta T)T^+ = TT^+ + TT^+\delta TT^+ = TT^+(I + \delta TT^+)$$

and $\overline{T}T^+(I + \delta TT^+)^{-1} = TT^+$. Hence $T = TT^+T = \overline{T}T^+(I + \delta TT^+)^{-1}T$. This means $R(T) \subseteq R(\overline{T})$. Thus $R(\overline{T}) = R(T)$ and

$$P^\perp_{R(\overline{T})} = P^\perp_{R(T)} = TT^+(TT^+)^*\{I - [TT^+ - (TT^+)^*]^2\}^{-1}.$$

By $R(I - TT^+) = N(T^+) = N(B)$, we have $T^+(I + \delta TT^+)^{-1}TT^+ = T^+(I + \delta TT^+)^{-1}$. Therefore by Lemma 2.5, we can get what we desired. □

Remark 2.11. It should be noted that if $N(T) \subseteq N(\delta T)$ or $R(\delta T) \subseteq R(T)$ holds, then the Hyers–Ulam stability constant $K_{\overline{T}}$ is continuous at T as δT is bounded and $\|\delta T\| \to 0$.

Recall that a closed operator $T : X \to Y$ is called left semi-Fredholm if $\dim N(T) < \infty$ and $R(T)$ is closed. It is called right semi-Fredholm if $\mathrm{codim}R(T) < \infty$ and $R(T)$ is closed. We say a closed operator T is semi-Fredholm if it is left or right semi-Fredholm. The following result involving semi-Fredholm operators holds.

Theorem 2.12. *Let X, Y be Hilbert spaces and let $T \in C(X,Y)$ be a semi-Fredholm operator. Let $T^+ \in B(Y,X)$ be a bounded generalized inverse of T and $\delta T \in L(X,Y)$ be T−bounded with constants a, b. If $a\|T^+\| + b\|TT^+\| < 1$ and*

$$either \ \dim N(\overline{T}) = \dim N(T) < \infty \quad or \quad \mathrm{codim}R(\overline{T}) = \mathrm{codim}R(T) < \infty,$$

then $\overline{T} = T + \delta T$ has the Hyers–Ulam stability and $K_{\overline{T}} = \|\overline{T}^\dagger\|$, where

$$\overline{T}^\dagger = \{I - [(T^+(I + \delta TT^+)^{-1}\overline{T})^{**} - (T^+(I + \delta TT^+)^{-1}(T + \delta T))^*]^2\}^{-1}$$
$$[T^+(I + \delta TT^+)^{-1}\overline{T}]^*T^+(I + \delta TT^+)^{-1}[\overline{T}T^+(I + \delta TT^+)^{-1}]^*$$
$$\{I - [\overline{T}T^+(I + \delta TT^+)^{-1} - (\overline{T}T^+(I + \delta TT^+)^{-1})^*]^2\}^{-1}.$$

Proof. By Lemma 2.6, $B = B\overline{T}B$ on Y, where $B = T^+(I+\delta TT^+)^{-1}$. Then $B\overline{T}B\overline{T} = B\overline{T}$ and it follows that $B\overline{T}$ is an idempotent on $D(\overline{T})$. Due to $\overline{T}(I - T^+T) = (T + \delta T)(I - T^+T) = \delta T(I - T^+T)$ and

$$
\begin{aligned}
\|[\delta T(I - T^+T)]x\| &\leq a\|(I - T^+T)x\| + b\|T(I - T^+T)x\| \\
&\leq a\|I - T^+T\| \cdot \|x\| \qquad (x \in D(T))
\end{aligned}
$$

we conclude that

$$
\begin{aligned}
B\overline{T} &= T^+[I + (\overline{T} - T)T^+]^{-1}\overline{T} \\
&= T^+[I + (\overline{T} - T)T^+]^{-1}\overline{T}T^+T + T^+[I + (\overline{T} - T)T^+]^{-1}\overline{T}(I - T^+T) \\
&= T^+T + T^+[I + (\overline{T} - T)T^+]^{-1}\overline{T}(I - T^+T)
\end{aligned}
$$

is bounded. Hence $B\overline{T}$ can be extended uniquely from $D(\overline{T})$ to X. We denote its extension by S, which is defined by $Sx = \lim_{n \to +\infty} B\overline{T}x_n$ for all $x \in X$, where $x_n \in D(\overline{T})$ satisfies $x_n \to x$. It is easy to verify $S \in B(X)$ and $S^2 = S$. Since $R(B\overline{T}) \subset R(S) \subset \overline{R(B\overline{T})}$, we get $R(S) = \overline{R(B\overline{T})} = \overline{R(B)} = \overline{R(T^+)}$. Next, for any $x \in N(S)$, then there exists $\{x_n\} \subset D(\overline{T})$ with $x_n \to x$ such that $B\overline{T}x_n \to Sx = 0$. Hence $x_n - B\overline{T}x_n \in N(B\overline{T})$ satisfies $x_n - B\overline{T}x_n \to x$, which implies $N(S) \subset \overline{N(B\overline{T})}$. Because $N(B\overline{T}) \subset N(S)$ and $N(S)$ is closed, $\overline{N(B\overline{T})} \subset N(S)$. Thus we conclude $N(S) = \overline{N(B\overline{T})}$. On the other hand, by the Closed Graph Theorem, we can observe that $\overline{T}B$ is a projector from Y onto $R(\overline{T}B)$ and $N(\overline{T}B) = N(B) = N(T^+)$. Thus by $X = R(S) \oplus N(S)$ and $Y = R(\overline{T}B) \oplus N(B)$, we get

$$
\overline{R(T^+)} \oplus N(T) = X = \overline{R(B\overline{T})} \oplus \overline{N(B\overline{T})} = \overline{R(T^+)} \oplus \overline{N(B\overline{T})} \qquad (2.2)
$$

and

$$
R(T) \oplus N(T^+) = Y = R(\overline{T}B) \oplus N(T^+) = R(\overline{T}) + N(T^+) = R(\overline{T}) \oplus N^-, \quad (2.3)
$$

where N^- satisfies $N(T^+) = N^- \oplus (R(\overline{T}) \cap N(T^+))$. If $\dim N(\overline{T}) = \dim N(T) < \infty$, then by (2.2), we get $\dim \overline{N(B\overline{T})} = \dim N(\overline{T})$. Noting $N(\overline{T}) \subseteq \overline{N(B\overline{T})}$, we can obtain $N(\overline{T}) = \overline{N(B\overline{T})}$. Hence for any $y \in R(\overline{T}) \cap N(T^+)$, there exists $x \in D(\overline{T})$ such that $y = \overline{T}x$ and $T^+\overline{T}x = 0$. This means $\overline{T}x \in N(T^+) = N(B)$. Thus $x \in N(B\overline{T})$ and so $x \in N(\overline{T})$, which implies $y = \overline{T}x = 0$. Therefore $R(\overline{T}) \cap N(T^+) = \{0\}$. If $\text{codim} R(\overline{T}) = \text{codim} R(T) < \infty$, then by (2.3), $\dim N(T^+) = \dim N^-$ and so $R(\overline{T}) \cap N(T^+) = \{0\}$. Using Theorem 2.7 we can get what we desired. $\qquad\square$

3. THE CASE OF BOUNDED LINEAR OPERATORS

In this section, we shall give some sufficient and necessary conditions for the Hyers–Ulam stability constants to be continuous.

Theorem 3.1. *Let X, Y be Hilbert spaces and let $T \in B(X, Y)$ have the Hyers–Ulam stability, i.e, T has a Moore–Penrose inverse $T^\dagger \in B(Y, X)$. If $\delta T \in B(X, Y)$ satisfies $\|\delta T\|\|T^\dagger\| < 1$, then the following statements are equivalent:*

(1) $B = T^\dagger(I + \delta T T^\dagger)^{-1} : Y \to X$ *is a bounded generalized inverse of* $\overline{T} = T + \delta T$;

(2) $(I + \delta T T^\dagger)^{-1} R(\overline{T}) = R(T)$;

(3) $(I + \delta T T^\dagger)^{-1}\overline{T}$ *maps* $N(T)$ *into* $R(T)$;

(4) $(I + T^\dagger \delta T)^{-1} N(T) = N(\overline{T})$;

(5) $R(\overline{T}) \cap N(T^\dagger) = \{0\}$;

(6) $\overline{T}$ *has the Hyers–Ulam stability and the Hyers–Ulam stability constant $K_{\overline{T}}$ satisfies*

$$\lim_{\|\delta T\| \to 0} K_{\overline{T}} = K_T;$$

(7) $\overline{T}$ *has the Hyers–Ulam stability and there exist $M > 0$ and $\varepsilon > 0$ such that $K_{\overline{T}} \leq M$ for all $\|\delta T\| < \varepsilon$;*

(8) $\overline{T}$ *has the bounded Moore–Penrose inverse $\overline{T}^\dagger \in B(Y, X)$ with $\displaystyle\lim_{\|\delta T\| \to 0} \overline{T}^\dagger = T^\dagger$.*

In this case, $K_{\overline{T}} = \|\overline{T}^\dagger\|$ and

$$
\begin{aligned}
\overline{T}^\dagger =\ & \{I - [T^+(I + \delta T T^+)^{-1}\overline{T} - (T^+(I + \delta T T^+)^{-1}\overline{T})^*]^2\}^{-1} \\
& [T^+(I + \delta T T^+)^{-1}\overline{T}]^* T^+(I + \delta T T^+)^{-1}[\overline{T} T^+(I + \delta T T^+)^{-1}]^* \\
& \{I - [\overline{T} T^+(I + \delta T T^+)^{-1} - (\overline{T} T^+(I + \delta T T^+)^{-1})^*]^2\}^{-1}.
\end{aligned}
$$

Proof. In Theorem 2.7, we take $a = \|\delta T\|$ and $b = 0$. Then it follows that (1), (2), (3) and (5) are equivalent. The equivalence of (4) and (5) can be found in [11]. By Theorem 2.2 and Theorem 2.7, we can see (5) $\Rightarrow$ (8) and (8) $\Rightarrow$ (6). Obviously, (6) $\Rightarrow$ (7). Next we shall show (7) $\Rightarrow$ (8) and (8) $\Rightarrow$ (5). Since

$$
\begin{aligned}
& \overline{T}^\dagger - T^\dagger \\
=\ & \overline{T}^\dagger(I - TT^\dagger) + \overline{T}^\dagger(T - \overline{T})T^\dagger + (\overline{T}^\dagger \overline{T} - I)T^\dagger \\
=\ & \overline{T}^\dagger \overline{T} \overline{T}^\dagger(I - TT^\dagger) + \overline{T}^\dagger(T - \overline{T})T^\dagger + (\overline{T}^\dagger \overline{T} - I)T^\dagger T T^\dagger \\
=\ & \overline{T}^\dagger[(I - TT^\dagger)\overline{T} \overline{T}^\dagger]^* + \overline{T}^\dagger(T - \overline{T})T^\dagger + [T^\dagger T(\overline{T}^\dagger \overline{T} - I)]^* T^\dagger \\
=\ & \overline{T}^\dagger[(I - TT^\dagger)(\overline{T} - T)\overline{T}^\dagger]^* + \overline{T}^\dagger(T - \overline{T})T^\dagger + [T^\dagger(T - \overline{T})(\overline{T}^\dagger \overline{T} - I)]^* T^\dagger \\
=\ & \overline{T}^\dagger(\overline{T}^\dagger)^*(\overline{T} - T)^*(I - TT^\dagger) + \overline{T}^\dagger(T - \overline{T})T^\dagger + (\overline{T}^\dagger \overline{T} - I)(T - \overline{T})^*(T^\dagger)^* T^\dagger,
\end{aligned}
$$

we can get

$$||\overline{T}^{\dagger} - T^{\dagger}|| \le (||\overline{T}^{\dagger}||^2 + ||\overline{T}^{\dagger}||||T^{\dagger}|| + ||T^{\dagger}||^2)||\delta T||.$$

Combining it with $||\overline{T}^{\dagger}|| = K_{\overline{T}} \le M$, we can conclude that $\lim\limits_{||\delta T|| \to 0} \overline{T}^{\dagger} = T^{\dagger}$. To the end, we only need to prove $(8) \Rightarrow (5)$. If $\overline{T}$ has the bounded Moore–Penrose inverse $\overline{T}^{\dagger}$ with $\lim\limits_{||\delta T|| \to 0} \overline{T}^{\dagger} = T^{\dagger}$, then $P_{N(\overline{T})}^{\perp} = I - \overline{T}^{\dagger}\overline{T}$ and $P_{N(T)}^{\perp} = I - T^{\dagger}T$ and $\lim\limits_{||\delta T|| \to 0} P_{N(\overline{T})}^{\perp} = P_{N(T)}^{\perp}$. Without loss of generality, we may assume $||P_{N(\overline{T})}^{\perp} - P_{N(T)}^{\perp}|| < 1$. Then

$$\begin{aligned} P_{N(T)}^{\perp}N(\overline{T}) &= (I - T^{\dagger}T)N(\overline{T}) \\ &= (I - T^{\dagger}\overline{T} + T^{\dagger}\delta T)N(\overline{T}) \\ &= (I + T^{\dagger}\delta T)N(\overline{T}) \end{aligned}$$

and by the Banach Lemma, the operator

$$W = I - P_{N(T)}^{\perp} + P_{N(\overline{T})}^{\perp}P_{N(T)}^{\perp} = I + (P_{N(\overline{T})}^{\perp} - P_{N(T)}^{\perp})P_{N(T)}^{\perp}$$

is invertible and its inverse $W^{-1} : X \to X$ is bounded. Take any $y \in R(\overline{T}) \cap N(T^{\dagger})$, then $y = \overline{T}x$ and $T^{\dagger}\overline{T}x = 0$, where $x \in X$. Hence

$$T(I + T^{\dagger}\delta T)x = T(I + T^{\dagger}\overline{T} - T^{\dagger}T)x = 0,$$

which implies $(I + T^{\dagger}\delta T)x \in N(T)$. Therefore

$$\begin{aligned} (I + T^{\dagger}\delta T)x &= P_{N(T)}^{\perp}(I + T^{\dagger}\delta T)x \\ &= P_{N(T)}^{\perp}WW^{-1}(I + T^{\dagger}\delta T)x \\ &= P_{N(T)}^{\perp}(I - P_{N(T)}^{\perp} + P_{N(\overline{T})}^{\perp}P_{N(T)}^{\perp})W^{-1}(I + T^{\dagger}\delta T)x \\ &= P_{N(T)}^{\perp}P_{N(\overline{T})}^{\perp}P_{N(T)}^{\perp}W^{-1}(I + T^{\dagger}\delta T)x \\ &\in P_{N(T)}^{\perp}N(\overline{T}) = (I + T^{\dagger}\delta T)N(\overline{T}). \end{aligned}$$

Since $T^{\dagger}\delta T + I$ is invertible, we get $y = \overline{T}x = 0$. Thus we obtain (5). $\square$

Theorem 3.2. *Let X, Y be Hilbert spaces, $T \in B(X,Y)$ be a finite rank operator and $T^{\dagger} \in B(Y,X)$ be a Moore–Penrose inverse of T. If $\delta T \in B(X,Y)$ satisfies $||\delta T||||T^{\dagger}|| < 1$, then $\overline{T} = T + \delta T$ has the Hyers–Ulam stability and the Hyers–Ulam stability constant $K_{\overline{T}}$ satisfies $\lim\limits_{||\delta T|| \to 0} K_{\overline{T}} = K_T$ if and only if*

$$\text{Rank } \overline{T} = \text{Rank } T < +\infty.$$

Proof. The necessity follows from $(6) \Rightarrow (2)$ in Theorem 3.1. Next, we shall show the sufficiency. By Lemma 2.5, $\overline{T}B$ is a projector from Y onto $R(\overline{T}B)$. Then

$$R(T) \dotplus N(T^\dagger) = Y = R(\overline{T}B) \oplus N(B) = R(\overline{T}B) \oplus N(T^\dagger).$$

Noting $\dim R(T) = \mathrm{codim} N(T^\dagger) = \dim R(\overline{T}B)$, we have $\dim R(\overline{T}) = \dim R(\overline{T}B)$, so $R(\overline{T}B) = R(\overline{T})$. Hence by $R(\overline{T}B) \cap N(T^\dagger) = R(\overline{T}B) \cap N(B) = \{0\}$, we get $R(\overline{T}) \cap N(T^\dagger) = \{0\}$. $\square$

Theorem 3.3. *Let X, Y be Hilbert spaces and let $T \in B(X,Y)$ be a semi-Fredholm operator. If T has the bounded Moore–Penrose inverse $T^\dagger \in B(Y,X)$ and $\delta T \in B(X,Y)$ satisfies $\|\delta T\| \|T^\dagger\| < 1$, then $\overline{T} = T + \delta T$ has the Hyers–Ulam stability and the Hyers–Ulam stability constant $K_{\overline{T}}$ satisfies $\lim\limits_{\|\delta T\| \to 0} K_{\overline{T}} = K_T$ if and only if*

$$either \quad \dim N(\overline{T}) = \dim N(T) < \infty \ or \ \mathrm{codim} R(\overline{T}) = \mathrm{codim} R(T) < \infty.$$

Proof. It is easy to see that the sufficiency follows from Theorem 2.12. Next we shall prove the necessity. If $\dim N(T) < \infty$, then by $(6) \Rightarrow (4)$ in Theorem 3.1, we can see $\dim N(\overline{T}) = \dim N(T)$. If $\mathrm{codim} R(T) < \infty$, then by Theorem 3.1, $T^\dagger(I + \delta T T^\dagger)^{-1} : Y \to X$ is a bounded generalized inverse of $T + \delta T$. Hence from Lemma 2.6, $N(B) = N(T^\dagger)$ and

$$Y = R(T) \dotplus N(T^\dagger) = R(\overline{T}) \oplus N(B) = R(\overline{T}) \oplus N(T^\dagger).$$

Thus $\dim N(T^\dagger) = \mathrm{codim} R(T) < \infty$ and therefore $\mathrm{codim} R(\overline{T}) = \mathrm{codim} R(T)$. $\square$

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[1] College of Mathematics, Yangzhou University, Yangzhou 225002, China; School of Mathematical Sciences, Monash University, VIC 3800, Australia

E-mail address: qlhmath@yahoo.com.cn

[2] Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran.

E-mail address: moslehian@um.ac.ir and moslehian@memebr.ams.org

URL: http://profsite.um.ac.ir/~moslehian/